

Quantile based modeling of diurnal temperature range with the five-parameter lambda distribution

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Abstract

Diurnal temperature range is an important variable in climate science that can provide information regarding climate variability and climate change. Changes in diurnal temperature range can have implications for hydrology, human health and ecology, among others. Yet, the statistical literature on modeling diurnal temperature range is lacking. In this article we propose to model the distribution of diurnal temperature range using the five-parameter lambda (FPL) distribution. Additionally, in order to model diurnal temperature range with explanatory variables, we propose a distributional quantile regression model that combines quantile regression with marginal modeling using the FPL distribution. Inference is performed using the method of quantiles. The models are fitted to 30 years of daily observations of diurnal temperature range from 112 weather stations in the southern part of Norway. The flexible FPL distribution shows great promise as a model for diurnal temperature range, and performs well against competing models. The distributional quantile regression model is fitted to diurnal temperature range data using geographic, orographic, and climatological explanatory variables. It performs well and captures much of the spatial variation in the distribution of diurnal temperature range in Norway.

KEYWORDS

distributional quantile regression, diurnal temperature range, five-parameter lambda distribution, method of quantiles

1 | INTRODUCTION

In this article we develop distributional models for diurnal temperature range, which is the difference between daily maximum and minimum temperature. The fourth IPCC assessment report identified diurnal temperature range as a key uncertainty factor (Solomon et al., 2007), and the fifth report described a substantial knowledge gap surrounding this climate variable (Stocker et al., 2013). While more effort has since been made to understand diurnal temperature range, the literature on it is still lacking (Masson-Delmotte et al., 2021; Sun et al., 2019; Thorne et al., 2016b; Thorne, Donat, et al., 2016). Diurnal temperature range can be used as an index of radiative forced climate change, and as a useful index for assessing the output of general circulation models (Braganza et al., 2004). Additionally, it has been shown that diurnal

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temperature range is linked to human health conditions such as the risk of influenza (Park et al., 2020), risk of stroke (Vered et al., 2020), and overall health and mortality (Cheng et al., 2014; Lim et al., 2015). Changes in diurnal temperature range can also be of large importance within ecology (Henry, 2007; Kovi et al., 2016; Peng et al., 2013; Vasseur et al., 2014) and hydrology (Hanssen-Bauer et al., 2016), among others. Despite these areas of usage, to the best of our knowledge, there do not exist any attempts at statistical modeling of the distribution of diurnal temperature range in the literature. In their recent work of reassessing changes in diurnal temperature range worldwide, Thorne et al. (2016a) conclude that there is “only *medium confidence* in the magnitude of reductions in diurnal temperature range since 1950” and that “there is *low confidence* in trends and multidecadal variability in diurnal temperature range prior to 1950.” Thus, more knowledge about diurnal temperature range is needed.

In this article we propose to model the marginal distribution of diurnal temperature range with the five-parameter lambda (FPL) distribution (Gilchrist, 2000). The FPL distribution is an extension of the four-parameter generalized lambda distribution (Ramberg & Schmeiser, 1974), itself an extension of Tukey’s three-parameter lambda distribution (Tukey, 1962). This family of distributions has seen infrequent use within the statistical literature. Some areas of usage for the FPL distribution have been income modeling (Tarsitano, 2004) and reliability analysis (Ahmadabadi et al., 2012; Nair et al., 2013). The FPL distribution is tightly linked to the generalized Pareto distribution (e.g. Coles, 2001), as its quantile function is equal to the difference between two generalized Pareto quantile functions (see Section 3.1). The generalized Pareto distribution is often used for estimating extremely large or small quantiles, and has been much used for modeling both the upper and lower tails of temperature distributions (e.g. Rohrbeck et al., 2021; Stein, 2021a). Thus, the FPL distribution is a natural choice for modeling the difference between daily maximum and daily minimum temperature if one considers these as extreme upper and lower quantiles in the daily temperature distribution.

The FPL distribution can be used for modeling the distribution of diurnal temperature range in locations with available observations. However, most locations do not contain any available temperature data. Thus, it is also of interest to model diurnal temperature range in locations without daily temperature observations, using a regression model. Most classical regression models focus on estimating the conditional mean of a distribution, given a set of explanatory variables. However, the distribution of diurnal temperature range is complex, and its variance, skewness, and kurtosis vary in space (see Section 2). Thus, a regression model for the mean would not provide enough information about diurnal temperature range to be of much use. An alternative to regression on the mean is quantile regression (Koenker, 2005), where one models a set of conditional quantiles given some explanatory variables. Quantile regression is based on fewer assumptions than mean regression, and it allows for more flexible modeling of complex distributions. However, it can often lead to quantile crossing, meaning that certain combinations of the explanatory variables lead to non-monotonic quantile functions (Bondell et al., 2010; Cannon, 2011; Rodrigues & Fan, 2017). Furthermore, quantile regression can only estimate a finite set of quantiles, and it does not lend an easy way of estimating distributional properties like moments. An alternative that has gained more popularity in recent years is distributional regression, where one attempts to model the entire conditional distribution given a set of explanatory variables (e.g. Henzi et al., 2020; Klein et al., 2015; Schlosser et al., 2019). As stated by Hothorn et al. (2014), this should be the ultimate goal of any regression analysis. However, most distributional regression models can be somewhat complex and computationally demanding.

Here, we propose a conceptually simple and highly parallelizable distributional regression model, based on a combination of quantile regression and marginal modeling with the FPL distribution. Parameter estimation for the FPL distribution is performed using the method of quantiles (e.g. Koenker, 2005), which is based on minimizing the distance between a quantile function and a set of estimated quantiles. A thorough description of this estimation method is presented, and it is compared to competing estimation methods for the FPL distribution, both in a simulation study, and using real temperature observations. In order to ease parameter interpretation and numerical inference, a novel reparametrization of the FPL distribution is developed. The marginal FPL model and the distributional quantile regression model are fitted to 30 years of daily temperature observations from 112 weather stations in southern Norway, including both coastal and inland stations over a large range of altitudes. In order to properly evaluate the model fits, a closed-form expression for the continuous ranked probability score (CRPS, Matheson & Winkler, 1976) with an FPL forecast distribution is developed. This new modeling framework provides a rigorous alternative to analyze diurnal temperature range and its observed variation in time and space compared to current empirical approaches (e.g. Shelton et al., 2021; Sun et al., 2021; Vinnarasi et al., 2017; Wang et al., 2017).

The remainder of the article is organized as follows. Section 2 introduces daily temperature data from the southern part of Norway, and associated explanatory variables. Section 3 provides a motivation for the choice of modeling diurnal temperature range with the FPL distribution, and presents some of the properties of the distribution. The distributional quantile regression model is also developed here. In Section 4, the method of quantiles and two other competing

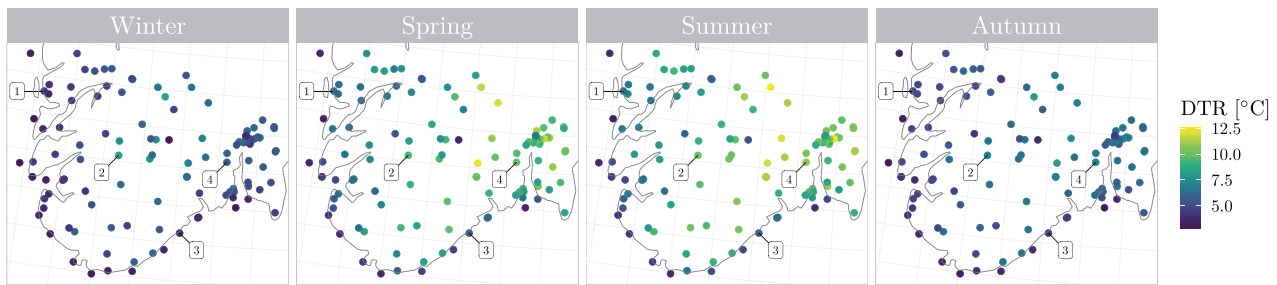


FIGURE 1 Empirical season medians of diurnal temperature range (DTR) at the 112 weather stations in our data set. The locations of the four stations from Figures 2, 6 and 8 are presented in all the plots. These are: (1) Flestrand, (2) Hovden-Lundane, (3) Lyngør fyr, and (4) Sande-Galleberg

inference methods are described, and a closed-form expression for the CRPS with an FPL forecast distribution is developed. In Section 5, a simulation study is performed, where we compare the method of quantiles with two competing inference methods. Finally, we apply our models to Norwegian diurnal temperature range data and evaluate the model fits in Section 6. The article concludes with a short discussion in Section 7.

2 | DATA

The analysis in this article is based on daily time series of air temperature observations from a set of weather stations in southern Norway. The data are openly available from Norwegian Meteorological Institute (2019). For each weather station, daily minimum and maximum temperatures between 18–18 UTC are used to find time series of diurnal temperature range. Data is downloaded from the 30 year time period from January 1, 1989 to December 31, 2018. Two thirds of the weather stations were already established in 1989, and the ages of the remaining stations are almost uniformly distributed between 1 and 30 years. Some stations are too recently established to be useful for our purposes, and others contain large amounts of missing data. Data cleaning is therefore performed by removing all weather stations that contains less than 180 observations from any of the four seasons of the year (winter: December–February; spring: March–May; summer: June–August; autumn: September–November). By cleaning the data we reduce the number of weather stations from an initial 133 to a new value of 112 stations. The locations of these are displayed in Figure 1 together with median diurnal temperature range for each season. These 112 stations span altitudes from 0 to 1900 m above sea level, and contains observations from tundra, subarctic, and oceanic climates (Kottek et al., 2006).

Our modeling framework does not specifically account for measurement and round-off errors. Obvious errors, resulting in negative diurnal temperature range, are removed from the data. Due to the existence of negative data, it is expected that there also are erroneous data among the positive range values. However, accounting for these errors is outside the scope of this article.

Six explanatory variables are used for modeling diurnal temperature range: easting, northing, distance to the open sea and altitude, in addition to the historical mean and variance of daily mean temperature at each location. This is estimated using the records of daily temperature observations from each station. Exploratory analysis finds evidence that the marginal distribution of diurnal temperature range can be approximated as being constant within each season (results not shown), but that it varies between the seasons. Historical mean and variance of the daily mean temperature are therefore computed for each season separately. The distance to the open sea is derived from a digital elevation model of Norway, with resolution $50 \times 50 \text{ m}^2$, published by the Norwegian Mapping Authority (<https://hoydedata.no>). Time series of daily mean temperature, along with longitude, latitude and altitude are freely available from Norwegian Meteorological Institute (2019). Easting and northing are based on UTM 32 coordinates.

Figure 1 shows the median of diurnal temperature range for each season and each weather station. It reveals clear seasonal and spatial patterns. In particular, the values appear higher during spring and summer than during winter and autumn. Similar patterns are also found when examining other quantiles. Histograms of diurnal temperature range for four selected stations and seasons are presented in Figure 2. We observe considerable differences in the shapes of the histograms.

To explore the relation between the explanatory variables and diurnal temperature range, we fit a simple linear model with different quantiles of diurnal temperature range against standardized versions of the explanatory variables. These

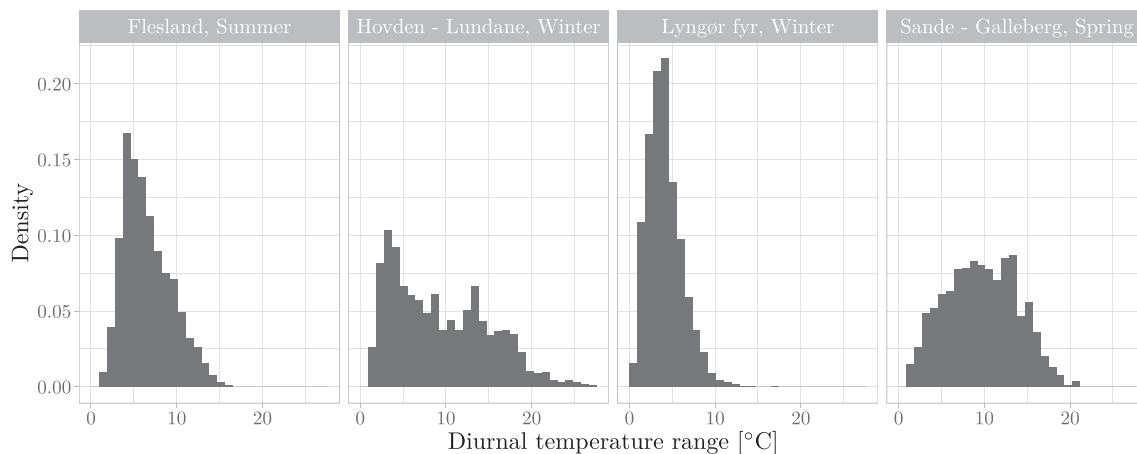


FIGURE 2 Histograms of diurnal temperature range for four selected weather stations and seasons

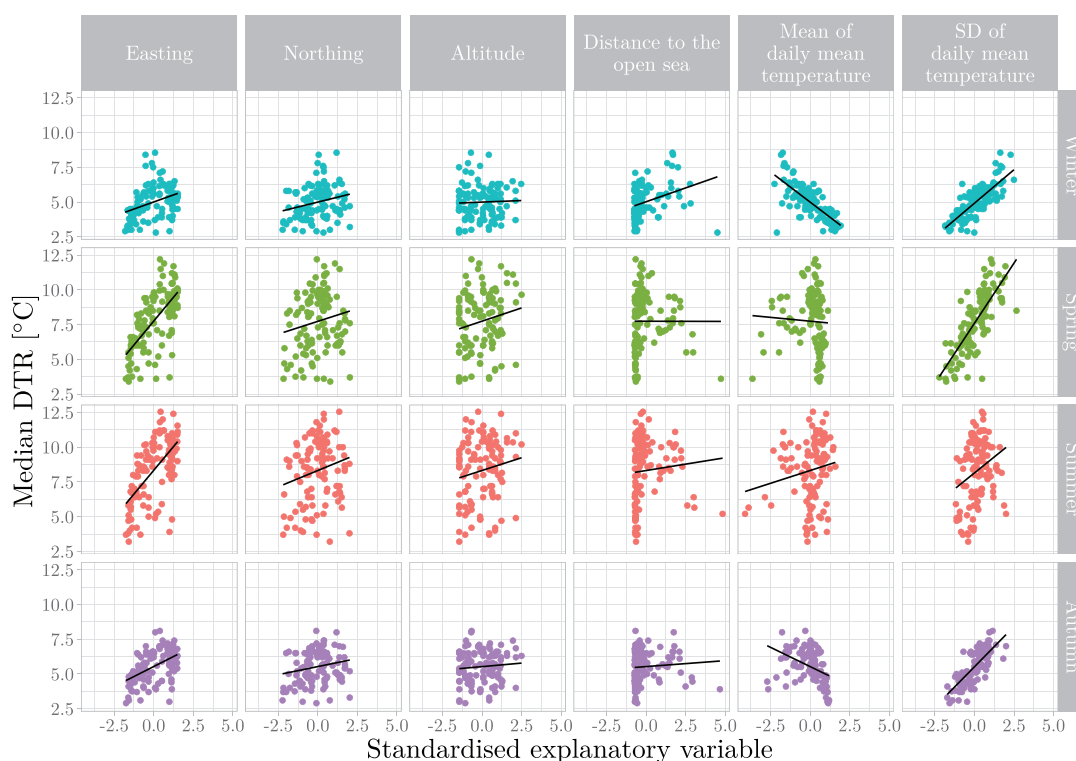


FIGURE 3 Linear relationships between the six explanatory variables and the median of diurnal temperature range (DTR) at the 112 weather stations in our data set. All explanatory variables are standardized to have zero mean and a standard deviation of one

linear model fits are presented in Figure 3 for the median of diurnal temperature range. Most of the estimated trends are significant at the 5%-level. Especially for the mean and variance of historical daily mean temperature, there is a strong linear relationship during winter and autumn. Similar trends are found for all other examined quantiles.

3 | MODELS

3.1 | Marginal modeling with the FPL distribution

We assume that daily minimum and maximum temperature can be described well by the generalized Pareto distribution. The generalized Pareto distribution is a common model for extreme observations (e.g. Coles, 2001), and it is often

used for modeling extreme temperature (e.g. Castro-Camilo et al., 2021; Davison & Huser, 2019). Additionally, Rohrbeck et al. (2021) model daily temperature in the Red Sea by assuming that both the upper and lower tails of daily temperature can be modeled using the generalized Pareto distribution, and Stein (2021b) proposes to model climatological phenomena using parametric distributions that behaves like the generalized Pareto distribution in both tails, and uses this approach to model daily average temperature near Calgary during winter. The generalized Pareto distribution can be described through its quantile function (e.g. Hosking & Wallis, 1987)

$$Q(p; \mu, \eta, \xi) = \inf\{x \in \mathbb{R} : p \leq F(x; \mu, \sigma, \xi)\} = \mu + \eta \begin{cases} (1 - (1 - p)^\xi) / \xi, & \xi \neq 0 \\ -\log(1 - p), & \xi = 0, \end{cases}$$

where F is the cumulative distribution function, and μ , η , and ξ acts as the location, scale, and shape parameters, respectively. Diurnal temperature range is equal to the difference between daily maximum and daily minimum temperature, and can therefore be modeled as the difference between the quantiles of two generalized Pareto distributions. We model maximum daily temperature as some quantile of a generalized Pareto distribution, while minimum daily temperature is modeled as some quantile of a reflected generalized Pareto distribution. If the random variable X has a quantile function $Q_X(p)$, then $-X$ has the quantile function $Q_{-X}(p) = -Q_X(1 - p)$. This results in an expression for the diurnal temperature range:

$$Q_{\text{range}} = Q_{\text{max}}(p_1; \mu_1, \eta_1, \xi_1) - Q_{\text{min}}(p_2; \mu_2, \eta_2, \xi_2) = \mu_1 + \frac{\eta_1}{\xi_1}(1 - (1 - p_1)^{\xi_1}) + \mu_2 + \frac{\eta_2}{\xi_2}(1 - p_2^{\xi_2}).$$

In the case of $\xi_1 = 0$ or $\xi_2 = 0$ the expression is simplified, since $\lim_{\lambda \rightarrow 0}(p^\lambda - 1)/\lambda = \log p$. We reparametrize by setting $p_1 = p$, $p_2 = p^a$, $\xi_2^* = a\xi_2$, and $\eta_2^* = a\eta_2$, where $a = \log p_2 / \log p$. This gives

$$Q_{\text{range}}(p) = \mu_1 + \frac{\eta_1}{\xi_1}(1 - (1 - p)^{\xi_1}) + \mu_2 + \frac{\eta_2^*}{\xi_2^*}(1 - p^{\xi_2^*}).$$

Further manipulation of the expression yields

$$Q_{\text{range}}(p) = (\mu_1 + \mu_2) + \frac{\eta_1 - \eta_2^*}{2} \left\{ \left(1 - \frac{\eta_1 + \eta_2^*}{\eta_1 - \eta_2^*}\right) \frac{p^{\xi_2^*} - 1}{\xi_2^*} - \left(1 + \frac{\eta_1 + \eta_2^*}{\eta_1 - \eta_2^*}\right) \frac{(1 - p)^{\xi_1} - 1}{\xi_1} \right\},$$

which is equal to the quantile function of a FPL distribution (Gilchrist, 2000),

$$Q(p; \lambda) = \lambda_1 + \frac{\lambda_2}{2} \left\{ (1 - \lambda_3) \frac{p^{\lambda_4} - 1}{\lambda_4} - (1 + \lambda_3) \frac{(1 - p)^{\lambda_5} - 1}{\lambda_5} \right\}, \quad \lambda_2 > 0, \lambda_3 \in [-1, 1], \quad (1)$$

Consequently, we expect the FPL distribution to be a suitable model for diurnal temperature range.

No analytic expression for the probability density function or cumulative distribution function of the FPL distribution exists, although the density for a given p can be obtained as the reciprocal of the quantile derivative, $(dQ(p; \lambda)/dp)^{-1}$. From the quantile function (1) of the FPL distribution it is clear that λ_1 acts as a location parameter and λ_2 as a scale parameter of the distribution. We notice that $\lambda_4 = \xi_2^*$ and $\lambda_5 = \xi_1$, which means that these two parameters control the behavior of the left and right tails, respectively. The final parameter λ_3 acts as a weight between the two tails.

The support of the FPL distribution can be both finite and infinite. This makes the distribution flexible for modeling a variety of different phenomena. The support is given by

$$[Q(0; \lambda), Q(1, \lambda)] = \lambda_1 + \frac{\lambda_2}{2} \begin{cases} \left[-\frac{1-\lambda_3}{\lambda_4}, \frac{1+\lambda_3}{\lambda_5} \right], & \lambda_4, \lambda_5 > 0 \\ \left[-\frac{1-\lambda_3}{\lambda_4}, \infty \right), & \lambda_4 > 0, \lambda_5 \leq 0 \\ \left(-\infty, \frac{1+\lambda_3}{\lambda_5} \right], & \lambda_4 \leq 0, \lambda_5 > 0 \end{cases}$$

Diurnal temperature range is always positive. In order to ensure a positive support for the FPL distribution, one must enforce the inequality-constraints

$$\lambda_1 - \frac{\lambda_2(1 - \lambda_3)}{2\lambda_4} > 0, \quad \lambda_4 > 0. \quad (2)$$

The parameterization in (1) is intuitive in the sense that it stems from the combination of two generalized Pareto distributions. However, for performing numerical parameter estimation, other representations are more appropriate. The location λ_1 and scale λ_2 are not clearly linked to any central moments or quantiles of the FPL distribution. We thus propose a reparametrization scheme with a new location parameter that is equal to the median of the FPL distribution, and a new scale parameter that is equal to the inter-quartile range of the FPL distribution,

$$\begin{aligned} \lambda_1^* &= Q(0.5; \lambda) = \lambda_1 + \frac{\lambda_2}{2} \left\{ (1 - \lambda_3) \frac{0.5^{\lambda_4} - 1}{\lambda_4} - (1 + \lambda_3) \frac{0.5^{\lambda_5} - 1}{\lambda_5} \right\}, \\ \lambda_2^* &= Q(0.75; \lambda) - Q(0.25; \lambda) = \frac{\lambda_2}{2} \left\{ \frac{1 - \lambda_3}{\lambda_4} (0.75^{\lambda_4} - 0.25^{\lambda_4}) + \frac{1 + \lambda_3}{\lambda_5} (0.75^{\lambda_5} - 0.25^{\lambda_5}) \right\}. \end{aligned} \quad (3)$$

The new parameter vector is denoted $\lambda^* = (\lambda_1^*, \lambda_2^*, \lambda_3, \lambda_4, \lambda_5)$. In order to simplify parameter constraints during any numerical estimation procedures, we further introduce the reparametrization

$$\tilde{\lambda}_1 = \lambda_1^*, \quad \tilde{\lambda}_2 = \log(e^{\lambda_2^*} - 1), \quad \tilde{\lambda}_3 = \log\left(\frac{1 - \lambda_3}{1 + \lambda_3}\right), \quad \tilde{\lambda}_4 = \log(e^{\lambda_4} - 1), \quad \tilde{\lambda}_5 = \log(e^{\lambda_5 + 0.5} - 1), \quad (4)$$

This results in an unconstrained parameter vector $\tilde{\lambda} \in \mathbb{R}^5$ and guarantees that $\lambda_2 > 0$, $\lambda_3 \in (-1, 1)$ and $\lambda_4 > 0$. We also restrict λ_5 to the interval $(-0.5, \infty)$, as this guarantees a finite mean and variance for the FPL distribution (e.g. Coles, 2001; Tarsitano, 2010). Exploratory data analysis (results not shown) finds that the right tail parameter in the diurnal temperature range distribution tends to be considerably larger than -0.5 . Davison and Huser (2019) model extreme temperatures in Spain with the generalized Pareto distribution and find that the tail parameter, which we denote by λ_5 , is approximately equal to 0.4. Castro-Camilo et al. (2021); Rohrbeck et al. (2021) model Red Sea temperatures with the generalized Pareto distribution and find that the tail parameter is larger than -0.1 . O'Sullivan et al. (2020) model temperature extremes in Dublin and find that the posterior median of the tail parameter is larger than 0.1. Based on these results and our exploratory data analysis, we are confident that the restriction of $\lambda_5 > -0.5$ should not lead to any loss in model performance.

The standard way of reparametrizing a parameter θ that is bounded away from zero is to set $\tilde{\theta} = \log \theta$. However, if θ attains a large value, a small error in the estimate for $\tilde{\theta}$ leads to a considerable error in the estimate for θ . The function $g(x) = \log(e^x - 1)$ has the property that $g(x) \approx \log x$ for small x and $g(x) \approx x$ for large x . This allows us to constrain the FPL parameters without risking large reparametrization instability because of an exponential relation between λ and $\tilde{\lambda}$.

The reparametrization to $\tilde{\lambda}$ eases numerical inference methods, and is used whenever we perform parameter estimation. However, the λ^* parametrization is more intuitive, and is therefore primarily used when describing our methods.

3.2 | Distributional quantile regression

We wish to model the marginal distribution of diurnal temperature range at locations without available temperature observations, using a regression model with explanatory variables. As described in Section 2, the distribution of diurnal temperature range is very rich. Many of its distributional properties seem to vary in space, and between seasons. Thus, it does not seem good enough to use for example, a generalized linear model (GLM) where we only allow the mean and variance of diurnal temperature range to vary in space. We suggest that one should apply a distributional regression model, where the entire distribution function is allowed to vary in space, (e.g. Henzi et al., 2020; Klein et al., 2015; Schlosser et al., 2019) for modeling diurnal temperature range. One way of performing distributional regression is to apply a latent Gaussian model with an FPL likelihood, such that all five FPL parameters are modeled as a linear combination of explanatory variables and Gaussian white noise. However, this leads to an unnecessarily complex model.

Additionally, as described in Section 5, the flexibility of the five FPL parameters might lead to something similar to identifiability problems, that can be problematic when we perform regression directly on the parameters and not on the distribution itself. Here, we propose a novel distributional regression model for diurnal temperature range which is based on combining quantile regression and marginal modeling with the FPL distribution, and we use this for modeling the marginal distribution of diurnal temperature range at locations with no temperature observations.

We first assume that any quantile in the distribution of diurnal temperature range can be modeled as a linear combination of explanatory variables. Let $y_i(\mathbf{s})$ be observation i of diurnal temperature range at location $\mathbf{s} \in S$, where S is the given study area. For any probability $p \in (0, 1)$, we assume that the diurnal temperature range can be modeled as

$$y_i(\mathbf{s}) = \mathbf{x}(\mathbf{s})^T \boldsymbol{\beta}_p + \varepsilon_{i,p}(\mathbf{s}), \quad i = 1, \dots, n(\mathbf{s}), \quad (5)$$

with explanatory variables $\mathbf{x}(\mathbf{s})$ and regression coefficients $\boldsymbol{\beta}_p$. The error terms $\varepsilon_{i,p}(\mathbf{s})$ are assumed to be independent and distributed such that $P(\varepsilon_{i,p}(\mathbf{s}) \leq 0) = p$ for all $\mathbf{s} \in S$ and $i = 1, \dots, n(\mathbf{s})$ (Koenker, 2005). We are now able to estimate quantiles $q_p(\mathbf{s}) = \mathbf{x}(\mathbf{s})^T \boldsymbol{\beta}_p$ of diurnal temperature range for any $p \in (0, 1)$, at any location \mathbf{s} with available explanatory variables $\mathbf{x}(\mathbf{s})$.

In order to turn this into a distributional regression model, we further propose to treat all $q_p(\mathbf{s})$ as quantiles of the FPL distribution at location \mathbf{s} . The only necessary assumption for performing quantile regression is that $P(y_i(\mathbf{s}) \leq \mathbf{x}(\mathbf{s})^T \boldsymbol{\beta}_p) = p$, and there need not be any disagreements between this and the assumption that the marginal distribution at any location \mathbf{s} is the FPL distribution. Consequently, we model the distribution of diurnal temperature range at location \mathbf{s} using the FPL distribution with the parameters λ^* that minimize the distance between $q_p(\mathbf{s})$ and the quantile function $Q(p; \lambda^*)$ of the FPL distribution (1). With this approach, we are able to describe the distribution of diurnal temperature range everywhere, using a parametric model. This makes it easier to interpret the distributional properties of diurnal temperature range than when we only use the semi-parametric quantile regression model.

A common problem with quantile regression is that the different estimated quantile models may cross, such that the estimator for $q_{p_i}(\mathbf{s})$ is larger than the estimator for $q_{p_j}(\mathbf{s})$ for $p_j > p_i$ (Bondell et al., 2010; Cannon, 2018; Rodrigues & Fan, 2017). However, by first performing quantile regression and then fitting the FPL distribution to the regression quantiles, this problem is easily fixed, as the FPL quantile function always is monotonic increasing. Consequently, there is no need to implement complicated quantile regression methods that ensure non-crossing quantiles, and we can base our modeling on the fast and simple regression model where each quantile is modeled separately.

For simplicity, the distributional quantile regression model is referred to as the regression model for the remainder of the article.

4 | INFERENCE

4.1 | Parameter estimation for the FPL distribution

We present three marginal parameter estimation methods for the FPL distribution: the method of quantiles, maximum likelihood estimation and the starship method.

4.1.1 | The method of quantiles

The method of quantiles is an estimation method similar to the better known method of moments, in which the distance between quantiles of a parametric distribution and empirical quantiles from observed data is minimized. Let y_1, \dots, y_n be independent and identically FPL distributed random variables with quantile function $Q(p; \lambda^*)$ and parameters λ^* . A set of m empirical quantiles $\hat{Q}(p_i) = \hat{q}_{p_i}$, $i = 1, 2, \dots, m$, are constructed from the observations \mathbf{y} . The method of quantiles estimator for λ^* , is found by minimizing the absolute distance

$$L(\lambda^*) = \sum_{i=1}^m \left| \hat{q}_{p_i} - Q(p_i; \lambda^*) \right|. \quad (6)$$

There do not exist any straightforward expressions for the probability density or the cumulative probability function of the FPL distribution. However, a simple and closed-form expression exists for its quantile function. This can make it more

natural to perform parameter estimation based on quantile matching instead of for example, likelihood-based estimation methods. In addition, Bignozzi et al. (2018) state that parameter estimation methods based on quantile matching can be preferable when distributions are heavy-tailed or their support varies with the parameters. Both of these conditions hold for the FPL distribution. Additionally, Bhatti et al. (2018) find that the method of quantiles outperforms both the method of moments and maximum likelihood estimation for parameter estimation under the Pareto distribution. As the FPL distribution can be described as the difference between two Pareto distributions, it should share some of the same properties. Tarsitano (2005) applies the method of quantiles for parameter estimation with the FPL distribution, using only five quantiles. He concludes that the method has several advantages, while a theoretical justification for the choice of quantiles is lacking. For a large set of observations $y_{(1)} \leq y_{(2)} \leq \dots \leq y_{(n)}$, the distribution function of $y_{(i)}$ is close to the empirical distribution function $\hat{F}(y_{(i)}) = (i - 0.5)/n$. Consequently, for a large set of n observations, we perform the method of quantiles by setting $p_i = (i - 0.5)/n$ and $\hat{q}_{p_i} = y_{(i)}$ for $i = 1, 2, \dots, n$, thus avoiding the issue of which quantiles to select. This is somewhat similar to the method of least absolute deviations by Tarsitano (2010). Koenker (2005) has shown that the method of quantiles estimator is consistent, provided that all estimated quantiles \hat{q} are consistent. Note that the order statistics of \mathbf{y} are dependent. This must be taken into account if one attempts to compute the variance of the estimator for λ^* .

A weakness of the method of quantiles is that it might return a parameter estimator $\hat{\lambda}^*$ such that $Q(0; \hat{\lambda}^*) > y_{(1)}$ or $Q(1; \hat{\lambda}^*) < y_{(n)}$. This problem is addressed by introducing inequality constraints when minimizing the loss function in (6), demanding that $Q(0; \lambda^*) < y_{(1)}$ and $y_{(n)} < Q(1; \lambda^*)$. In order to guarantee a positive support, the inequality constraints from (2) can also be enforced. Note that $\lambda_4 > 0$ is automatically enforced by optimizing over the $\tilde{\lambda}$ parametrization. The positive support constraint is slightly relaxed by only demanding that $Q(10^{-4}; \lambda^*) > 0$, as we find that this can considerably improve the model fit in certain cases. These constraints are enforced by performing numerical optimization using an augmented Lagrangian formulation (e.g. Nocedal & Wright, 2006), implemented within the R package `nloptr` (Birgin & Martínez, 2008; Johnson, 2020). Closed-form expressions are available both for the quantile loss function and for its gradient. However, in practice we find that the method of quantiles performs better when not including gradient information in the optimizer. Consequently, we minimize the augmented Lagrangian using the derivative-free Nelder-Mead algorithm (Nelder & Mead, 1965), also implemented in the `nloptr` package.

Due to the flexibility of the FPL distribution, the quantile loss (6) proves to be difficult to minimize without a good initial value for the FPL parameters. We utilize the connection between $(\lambda_1^*, \lambda_2^*)$ and the quantiles of the FPL distribution by setting the initial values equal to the empirical median and inter-quantile range of \mathbf{y} , respectively. Initial values for the remaining three parameters are selected using a quick grid search. We compute the quantile loss function for all combinations of $\lambda_3 \in \{-0.5, -0.25, 0, 0.25, 0.5\}$, $\lambda_4 \in \{0.1, 0.2, 0.4, 0.8, 1, 1.5\}$, and $\lambda_5 \in \{-0.4, -0.1, 0.1, 0.2, 0.4, 0.8, 1, 1.5\}$ and select the combination of parameters that minimizes it as initial values.

4.1.2 | Maximum likelihood estimation

No closed-form expression exists for the cumulative distribution function of the FPL distribution. However, as mentioned in Section 3.1, for a given probability p , the probability density function of the FPL distribution can be obtained as the reciprocal of the quantile derivative

$$f(y; \lambda) = \frac{2}{\lambda_2} \left\{ (1 - \lambda_3)p^{\lambda_4 - 1} + (1 + \lambda_3)(1 - p)^{\lambda_5 - 1} \right\}^{-1},$$

with $p = F(y; \lambda)$. A numerical approximation for the cumulative distribution function of the FPL distribution is available from the R package `gld` (King et al., 2020). For $\mathbf{y} = (y_1, \dots, y_n)^T$ this gives rise to the log-likelihood

$$\ell(\lambda; \mathbf{y}) = n \log 2 - n \log \lambda_2 - \sum_{i=1}^n \log \left\{ (1 - \lambda_3)p(y_i)^{\lambda_4 - 1} + (1 + \lambda_3)(1 - p(y_i))^{\lambda_5 - 1} \right\}.$$

A straightforward expression for the gradient of the log-likelihood cannot be provided, as it requires computing the derivative of $F(y; \lambda)$ with respect to λ . Maximization of the log-likelihood is performed using the same grid-search and augmented Lagrangian as for the method of quantiles, where we include the same inequality constraint to guarantee a positive support.

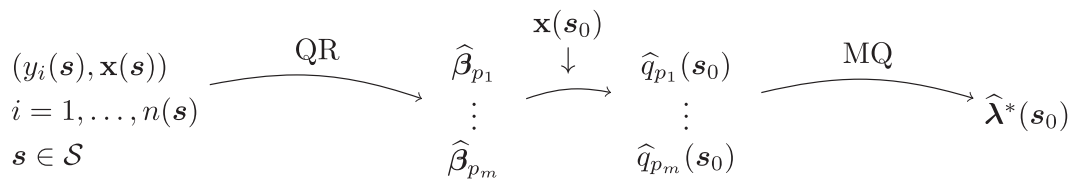


FIGURE 4 Diagram of the regression model. First, quantile regression (QR) is performed for modeling different quantiles of diurnal temperature range $y_i(\mathbf{s})$ given the explanatory variables $\mathbf{x}(\mathbf{s})$. Using the quantile regression model, the conditional quantiles $\hat{q}_{p_1}(\mathbf{s}_0), \hat{q}_{p_2}(\mathbf{s}_0) \dots, \hat{q}_{p_m}(\mathbf{s}_0)$ are estimated, given the explanatory variables $\mathbf{x}(\mathbf{s}_0)$ at a specific location. Then, using the method of quantiles (MQ), the FPL distribution is fitted to the conditional quantiles, resulting in the estimator $\hat{\lambda}^*(\mathbf{s}_0)$ for the FPL parameters

4.1.3 | The starship method

A straightforward way of modeling with the FPL distribution is to use the already implemented functions in the R package `gld` (King et al., 2020). This package is mostly focused on the four-parameter generalized lambda distribution, but it also includes one inference method for the FPL distribution, namely the starship method (King & MacGillivray, 1999; Owen, 1988). The starship method is based on the fact that if \mathbf{y} has distribution function $F(\cdot; \lambda^*)$, the transformed variable $\mathbf{u} = F(\mathbf{y}; \lambda^*)$ has a uniform distribution. Parameters can therefore be estimated by minimizing any goodness-of-fit statistic between the uniform distribution and $F(\mathbf{y}; \lambda^*)$.

The starship suffers from the same problems as the maximum likelihood estimator, namely that no closed-form expression exists for the cumulative distribution function of the FPL distribution, which therefore has to be numerically approximated. The implemented method in the `gld` package performs minimization using the Anderson-Darling statistic. However, the `gld` implementation of the starship method is very computationally inefficient, so in order to compare it with the previously described methods for large amounts of data, we implement our own version of the starship, based on the `gld` implementation. This implementation performs minimization of the Anderson-Darling statistic using the same optimization approach as in the previous methods. On a sample of 10^4 observations, our version of the starship estimates λ^* in approximately 50 s, while the `gld` implementation uses approximately 330 s. The two implementations seem to perform equally well numerically.

4.2 | Parameter estimation in the regression model

Inference for the regression model is divided into two steps. The estimation procedure is illustrated in Figure 4. First, quantile regression is performed using the R package `quantreg` (Koenker, 2018), separately for each of the probabilities $p_i = i/100$, $i = 1, 2, \dots, 99$. Then, at any location \mathbf{s}_0 with available explanatory variables where we wish to model diurnal temperature range, we estimate the conditional quantiles $\hat{q}_{p_i}(\mathbf{s}_0)$. The quantile function of the FPL distribution is then fitted to the 99 estimated quantiles $\hat{q}_{p_1}(\mathbf{s}_0), \hat{q}_{p_2}(\mathbf{s}_0) \dots, \hat{q}_{p_{99}}(\mathbf{s}_0)$ using a marginal parameter estimation method. There are no good ways of extending the maximum likelihood or starship method to fit a quantile function to a set of quantiles. However, the method of quantiles is perfect for this kind of parameter estimation problem. Consequently, the method of quantiles is used for fitting the FPL distribution to the 99 estimated quantiles, resulting in an estimator $\hat{\lambda}^*(\mathbf{s}_0)$ for the FPL parameters at \mathbf{s}_0 . All 99 quantiles are modeled independently of each other, meaning that numerical inference can be executed in parallel. Fitting the FPL distribution to estimated quantiles at different locations is also an independent operation that can be performed in parallel. The proposed regression model is therefore highly parallelizable.

4.3 | Model evaluation

Model performance is evaluated using the continuous ranked probability score (CRPS Gneiting & Raftery, 2007; Matheson & Winkler, 1976). Given a forecast distribution F and an observation y , the CRPS is equal to

$$S(F, y) = \int_{-\infty}^{\infty} (F(t) - I(t \geq y))^2 dt = 2 \int_0^1 \rho_p(y - F^{-1}(p)) dp, \quad (7)$$

with $\rho_p(u) = pu - I(u < 0)u$, where $I(\cdot)$ is an indicator function. This is a strictly proper scoring rule, meaning that if y has distribution function G , then $E[S(G, y)] \leq E[S(F, y)]$ for all forecast distributions F , with equality only if $F = G$. Due to scarce usage of the FPL distribution in the literature, to the best of our knowledge, a closed-form expression for the CRPS with an FPL forecast distribution has not yet been provided. Consequently, we derive the expression for the CRPS with an FPL forecast distribution. When modeling diurnal temperature range with the FPL distribution, the quantile formulation of the CRPS is especially useful. Given an FPL forecast distribution F with quantile function Q , the CRPS can be expressed as

$$S(F, y) = 2 \int_0^1 \rho_p(y - Q(p)) \, dp = y(2F(y) - 1) - 2 \int_0^1 pQ(p) \, dp + 2 \int_{F(y)}^1 Q(p) \, dp.$$

Both of these integrals are fairly straight-forward to compute with the FPL quantile function, as they are simply polynomials in p . Solving the first integral yields

$$\int_{F(y)}^1 Q(p) \, dp = (1 - F(y))\lambda_1 + \frac{\lambda_2}{2} \left((1 - \lambda_3) \left(\frac{1 - F(y)^{\lambda_4+1}}{\lambda_4(\lambda_4 + 1)} - \frac{1 - F(y)}{\lambda_4} \right) - (1 + \lambda_3) \left(\frac{(1 - F(y))^{\lambda_5+1}}{\lambda_5(\lambda_5 + 1)} - \frac{1 - F(y)}{\lambda_5} \right) \right),$$

while solving the second integral yields

$$\int_0^1 pQ(p) \, dp = \frac{\lambda_1}{2} + \frac{\lambda_2}{2} \left(-(1 - \lambda_3) \frac{1}{2(\lambda_4 + 2)} + (1 + \lambda_3) \frac{\lambda_5 + 3}{2(\lambda_5 + 1)(\lambda_5 + 2)} \right).$$

Thus, by numerically approximating $F(y)$ we are able to estimate the CRPS of the FPL distribution. In the special case of $\lambda_4 = 0$ or $\lambda_5 = 0$, the integrals are solved by using that

$$\begin{aligned} \int \log p \, dp &= p(\log p - 1), & \int \log(1 - p) \, dp &= (p - 1) \log(1 - p) - p, \\ \int p \log p \, dp &= \frac{1}{2} p^2 (2 \log p - 1), & \int p \log(1 - p) \, dp &= \frac{1}{4} (2(p^2 - 1) \log(1 - p) - p(p + 2)). \end{aligned}$$

The CRPS can be used for comparing competing forecasts. Given two forecasts F_1 and F_2 , and observations $\mathbf{y} = (y_1, \dots, y_n)^T$, we can compute the mean CRPS

$$S(F, \mathbf{y}) = \frac{1}{n} \sum_{i=1}^n S(F, y_i),$$

and choose the forecast with the lowest mean CRPS. However, the mean CRPS in itself does not provide much information about the goodness of fit of a forecast. Given a forecast F and observations \mathbf{y} with unknown distribution function G , there is no way of knowing if there is a large difference between $S(F, \mathbf{y})$ and $S(G, \mathbf{y})$, since G is unknown. Thus, we also evaluate model performance by studying quantile-quantile-plots (QQ-plots) and the probability integral transform (PIT).

If a random variable y has distribution function F , then the transformed variable $u = F(y)$ is uniformly distributed between zero and one. Given a forecast distribution F and observations \mathbf{y} one can therefore examine deviations between the distribution of $\mathbf{u} = F(\mathbf{y})$ and the standard uniform distribution. Heinrich et al. (2020) propose to evaluate model fit by examining the first two moments of \mathbf{u} . Denote the error in the first moment as $e_\mu = E(\mathbf{u}) - 0.5$ and the error in the second central moment as $e_\sigma = SD(\mathbf{u}) - 1/\sqrt{12}$. It follows that $e_\mu < 0$ indicates a positive bias and $e_\mu > 0$ indicates a negative bias. If $e_\sigma < 0$, the forecast distribution F is overdispersive, and if $e_\sigma > 0$, it is underdispersive.

5 | SIMULATION STUDY

5.1 | Setup

Simulation studies are performed to compare the different parameter estimation methods for the FPL distribution. We draw 500 random sets of FPL parameters λ^* . These are sampled such that they are of approximately the same magnitude

Algorithm 1. Sampling λ^*

```

while TRUE do
  Sample  $\lambda_1^* \sim \mathcal{N}(5, 3^2)$ 
  Sample  $\lambda_2^* \sim U(1.5, 8)$ 
  Sample  $\lambda_3 \sim U(-0.9, 0.9)$ 
  Sample  $\lambda_4 \sim U(0.01, 0.9)$ 
  Sample  $\lambda_5 \sim U(-0.3, 0.7)$ 
  if  $Q(0, \lambda^*) > 0$  then
    if  $Q(1, \lambda^*) - Q(0, \lambda^*) > 1$  then
      break
    end if
  end if
end while
return  $\lambda^*$ 

```

▷ Ensure a positive support
 ▷ Ensure that the support doesnot become too compact

as the estimated FPL parameters for diurnal temperature range in Section 6 (see Table 4), while also ensuring that we have a positive and wide enough support. The exact sampling scheme is given in Algorithm 1, with $\mathcal{N}(\mu, \sigma^2)$ denoting a Gaussian distribution with mean μ and variance σ^2 and $U(a, b)$ denoting a uniform distribution with limits a and b . For each set of FPL parameters we then sample n realizations from the $FPL(\lambda^*)$ distribution with $n = 2^i$ for $i = 7, 8, \dots, 14$. The parameter estimation methods described in Section 4.1 are then applied for estimating λ^* . We evaluate the overall fit to data and the ability of recovering the true parameter values. Overall model fit to data is evaluated using the CRPS. Since the true value of λ^* is known, we can compute the skill score $1 - CRPS(F, G)/CRPS(F, y)$, where F is our estimated distribution, G is the correct distribution and $CRPS(F, G)$ is the expected value of the CRPS with respect to G . Thus, a perfect forecast gives a skill score of zero, while all other forecasts give a skill score larger than zero and smaller than one. The ability to recovery the true values of λ^* is evaluated using the mean square error (MSE) between the true parameters λ^* and the estimated parameters $\hat{\lambda}^*$, over all 500 repetitions,

$$MSE(\lambda^*, \lambda) = \frac{1}{500} \sum_{i=1}^{500} \frac{1}{5} \sum_{j=1}^5 \left(\lambda_j^{*(i)} - \lambda_j^{(i)} \right)^2,$$

where $\lambda^{(i)} = (\lambda_1^{(i)}, \dots, \lambda_5^{(i)})^T$ are the true FPL parameters in simulation number i out of 500, and $\lambda^{*(i)}$ are the corresponding estimated parameters.

5.2 | Results

Table 1 displays the skill score, MSE and computation time for all methods. As n grows, the computation times for the maximum likelihood and starship methods grow considerably faster than the time for the method of quantiles. This happens because the method of quantiles is based solely on analytical expressions, while the other two methods require numerical estimation of the likelihood or distribution function of the FPL distribution. The method of quantiles has a worse skill score for small to medium sample sizes and a slightly better skill score for large sample sizes, whereas the starship method attains the worst skill score for large sample sizes. Interestingly, the method of quantiles fails to recover the correct FPL parameters, and has a much larger MSE than the other two methods. This demonstrates the flexibility of the FPL distribution, as we are able to achieve a better CRPS using the “wrong” parameter estimates. A closer examination of the estimated parameters finds that the large increase in MSE is caused almost solely by too large estimates of λ_4 and λ_5 . In some situations it seems that a large increase in λ_4 combined with a decrease in λ_3 yields almost no change in the overall shape of the FPL distribution. This makes sense, as increasing λ_4 leads to a thinner left tail, while decreasing λ_3 places more weight on the left tail. Similarly, a large increase in λ_5 can be mitigated by increasing λ_3 . When modeling diurnal temperature range, model fit is much more important than parameter recovery, as the FPL model is merely an assumption, and possibly not the true underlying distribution of diurnal temperature range. Thus, the low skill score and

TABLE 1 Mean skill scores, MSE and computation times for the method of quantiles (MQ), maximum likelihood (ML), and starship method, when performing parameter estimation 500 times on n samples drawn from an FPL distribution

	Method	$n = 2^7$	$n = 2^8$	$n = 2^9$	$n = 2^{10}$	$n = 2^{11}$	$n = 2^{12}$	$n = 2^{13}$	$n = 2^{14}$
Skill score ($\cdot 10^3$)	ML	6.12	3.18	1.48	0.73	0.40	0.21	0.11	0.09
	MQ	6.42	3.33	1.58	0.81	0.43	0.21	0.10	0.06
	Starship	6.12	3.18	1.49	0.75	0.40	0.26	0.20	0.21
MSE	ML	0.17	0.14	0.06	0.03	0.02	0.02	0.02	0.01
	MQ	14.01	207.16	201.66	6.87	32.20	3.85	5.04	0.47
	Starship	1.27	0.35	0.28	0.26	0.15	0.02	0.02	0.03
Time (s)	ML	0.6	0.9	1.5	3.0	5.6	11.0	22.6	46.4
	MQ	0.2	0.2	0.3	0.3	0.4	0.6	1.1	1.9
	Starship	0.8	1.1	1.8	3.3	6.5	14.1	30.4	63.9

Note: The skill scores are multiplied by 10^3 to get more readable results. Computation times are reported on a 2.4 GHz computation server.

fast computation times of the method of quantiles for large sample sizes make up for the fact that we seem to lose the ability to always recover the true parameters.

6 | MODELING DIURNAL TEMPERATURE RANGE IN SOUTHERN NORWAY

Diurnal temperature range in southern Norway is modeled separately for all seasons, using our two proposed models. Model calibration is evaluated using QQ-plots and the PIT. The marginal model is compared against several competing models using the CRPS. The regression model is tested in a leave-one-out cross-validation study, and the results are compared with the marginal model fits.

6.1 | Marginal modeling

In order to evaluate the model performance of the FPL distribution for diurnal temperature range, the distribution is used for modeling data from southern Norway. Diurnal temperature range is modeled separately for all seasons and weather stations, using the FPL distribution and three other competing parametric distributions: the gamma distribution, lognormal distribution and the generalized lambda (GL) distribution. The gamma- and lognormal distributions are fitted to data using maximum likelihood estimation, implemented in the R package MASS (Venables & Ripley, 2002). The GL distribution is a specialization of the FPL distribution, parametrized with four parameters, and it is fitted using maximum likelihood, implemented in the `gld` package. The FPL distribution is fitted to diurnal temperature range using all three inference methods described in Section 4. In-sample model comparison is performed using the CRPS. This is computed numerically for the GL distribution, using (7). For the gamma- and lognormal distributions, CRPS is computed using the `scoringRules` package (Jordan et al., 2019).

Table 2 displays the mean CRPS over all 112 weather stations for each season and all our chosen models. Apart from the FPL results during summer, all three model fits with the FPL distribution attain a lower mean CRPS than the competing models. The differences in CRPS might seem small, but a simple permutation test shows that there is a statistically significant difference between the scores of the FPL and GL distributions, except during summer when inference is performed with the method of quantiles. The same permutation test finds no evidence that there is a difference in CRPS when using the method of quantiles and the starship method, but there is some evidence that both the starship and the method of quantiles attains better model fits than the maximum likelihood estimation. The mean computation time for estimating the FPL parameters at a single location is approximately 0.3 s with the method of quantiles, 2.7 s with maximum likelihood estimation and 4.0 s with the starship method, meaning that it takes 2 min to estimate parameters for all stations and seasons with the method of quantiles, and 30 min with the starship method. Figure 5 displays the fitted

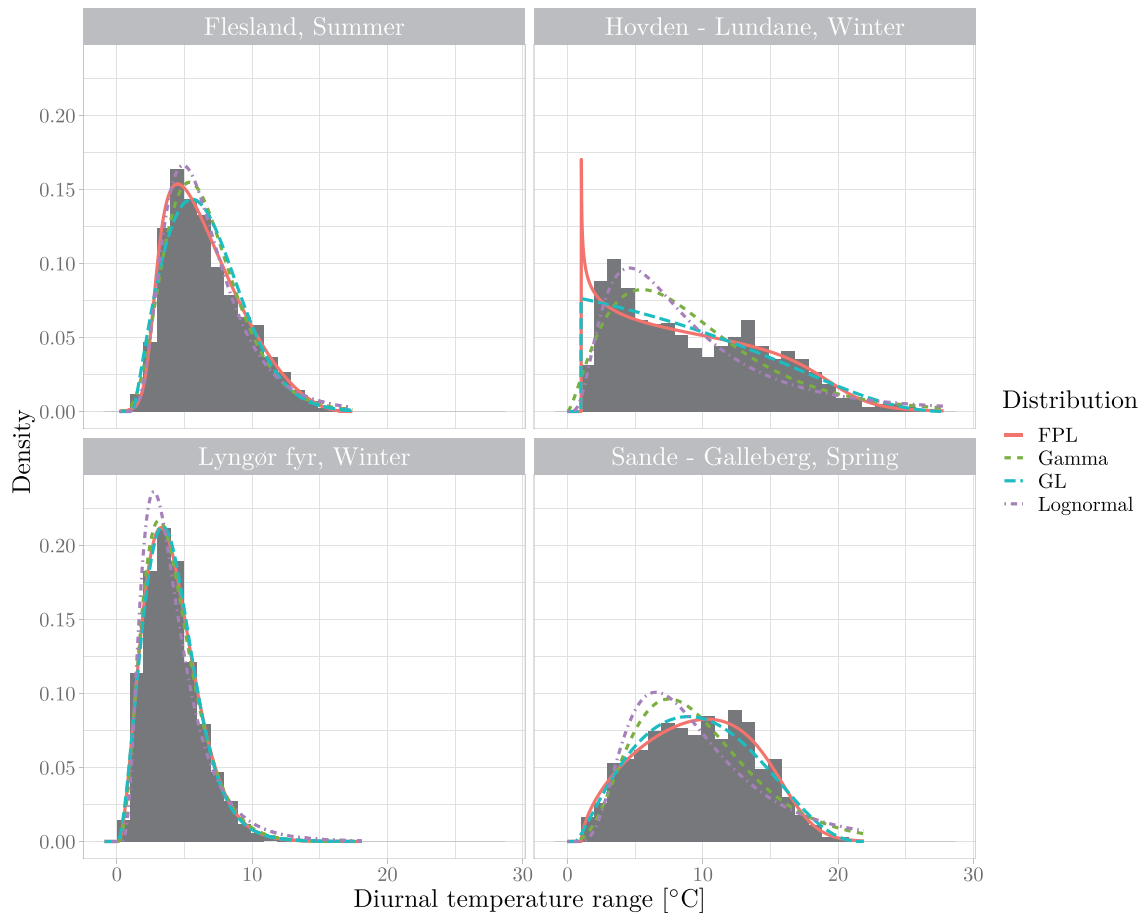


FIGURE 5 Estimated probability density functions using four different parametric distribution for modeling diurnal temperature range. The observations of diurnal temperature range are displayed using histograms

TABLE 2 Mean CRPS over all 112 weather stations during each season

Season	FPL (MQ)	FPL (ML)	FPL (starship)	GL	Gamma	Lognormal
Winter	1.577	1.577	1.577	1.578	1.580	1.589
Spring	2.109	2.109	2.108	2.111	2.122	2.141
Summer	1.851	1.849	1.849	1.851	1.857	1.867
Autumn	1.715	1.715	1.715	1.717	1.720	1.732

Note: Four different distributions are fitted to diurnal temperature range data. The FPL distribution are fitted to data using the method of quantiles (MQ), maximum likelihood (ML) estimation, and the starship method. The best mean CRPS for each season is written in bold.

probability density functions of all models at the four stations from Figure 2. The FPL distribution is fitted to data using the method of quantiles. It seems that the flexibility of the FPL distribution makes it able to model the many shapes of diurnal temperature range better than the competing models. Especially in the lower right plot one can see how the added flexibility of the FPL distribution allows it to provide a slightly better fit to data than the GL distribution, which clearly is the strongest competitor. The FPL distribution attains a lower CRPS than the competing models in all four sub-plots.

The FPL model fits are further evaluated to assess absolute performance. For the remainder of the article we choose to use the method of quantiles for fitting the FPL distribution to diurnal temperature range data, as all three inference methods perform almost equally well, and the method of quantiles is necessary for the regression model described in Section 3.2. Figure 6 displays different properties of the model fit of the FPL distribution at the four stations from Figure 5. It is evident that the FPL distribution provides a good model fit to observed diurnal temperature range data. The model struggles slightly with the bimodal distribution at Hovden-Lundane, but has an excellent fit to the data at the other three

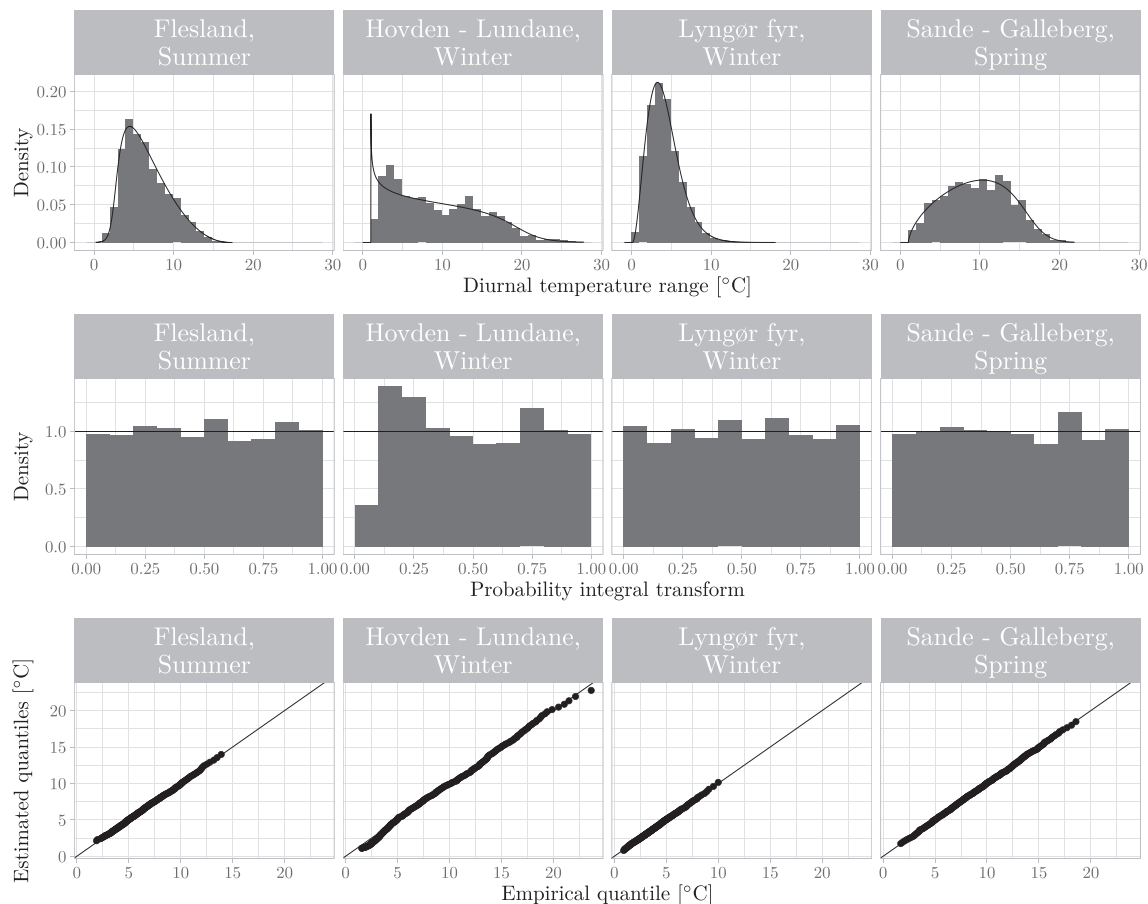


FIGURE 6 Upper plots: Histograms displaying observed diurnal temperature range are plotted along with the probability density functions of the FPL distribution. Middle plots: Histograms displaying the PIT of diurnal temperature range with the estimated FPL parameters. Lower plots: QQ-plots displaying sample quantiles of diurnal temperature range against distributional quantiles of the estimated FPL distributions

stations. A visual assessment of the model fit for all other stations finds that the results in Figure 6 are representative for most of the available data. The mean PIT errors e_μ and e_σ are displayed in Table 3. Both are of magnitude 10^{-4} for the marginal FPL model, implying high overall performance.

An overview of the estimated FPL parameters is given in Table 4. The location and scale parameters are largest during summer and spring, and the tail parameters λ_4 and λ_5 seem to take approximately the same values for all seasons. The estimator for λ_5 is mostly far away from -0.5 , indicating that the restriction of $\lambda_5 > -0.5$ has not lead to a decrease in model performance. The tail weight λ_3 seems to be almost evenly distributed between -1 and 1 , but its distribution is clearly most focused on the positive side, where it lends most weight to the right tail of the FPL distribution. When examining the marginal parameter estimates in a map (results not shown) we find that both $\hat{\lambda}_1^*$ and $\hat{\lambda}_4$ increase when moving eastwards. The opposite is found for $\hat{\lambda}_3$, which attains its largest values to the west. $\hat{\lambda}_2^*$ and $\hat{\lambda}_5$ take on low values along the coast, and increase as we move further away from the sea, and further to the east. There are some locations where the estimates for λ_4 and λ_5 are much larger than 1 . This is most likely caused by the problems discussed in Section 5, where a large change in a tail parameter combined with a change in λ_3 results in little change in the overall shape of the FPL distribution. The estimators for λ_2^* and λ_3 are also showing unusual values at these locations.

6.2 | Regression model

The regression model is applied for modeling diurnal temperature range for each season separately, using all the explanatory variables introduced in Section 2. Before we apply the regression model, each explanatory covariate is standardized

TABLE 3 Mean CRPS and PIT errors e_μ and e_σ over all 112 weather stations, computed for the marginal model and the regression model

Season	Model	CRPS	e_μ	e_σ
Winter	Marginal	1.58	$0.00 \cdot 10^{-2}$	$0.00 \cdot 10^{-2}$
	Regression, in-sample	1.63	$1.46 \cdot 10^{-2}$	$-0.30 \cdot 10^{-2}$
	Regression, out-of-sample	1.63	$1.44 \cdot 10^{-2}$	$-0.35 \cdot 10^{-2}$
Spring	Marginal	2.11	$0.06 \cdot 10^{-2}$	$-0.05 \cdot 10^{-2}$
	Regression, in-sample	2.29	$1.10 \cdot 10^{-2}$	$-1.35 \cdot 10^{-2}$
	Regression, out-of-sample	2.31	$1.08 \cdot 10^{-2}$	$-1.49 \cdot 10^{-2}$
Summer	Marginal	1.85	$0.01 \cdot 10^{-2}$	$0.01 \cdot 10^{-2}$
	Regression, in-sample	2.12	$1.70 \cdot 10^{-2}$	$-3.51 \cdot 10^{-2}$
	Regression, out-of-sample	2.16	$1.58 \cdot 10^{-2}$	$-3.74 \cdot 10^{-2}$
Autumn	Marginal	1.71	$0.03 \cdot 10^{-2}$	$-0.05 \cdot 10^{-2}$
	Regression, in-sample	1.76	$0.97 \cdot 10^{-2}$	$-0.54 \cdot 10^{-2}$
	Regression, out-of-sample	1.77	$0.94 \cdot 10^{-2}$	$-0.60 \cdot 10^{-2}$

Note: The regression model is fitted to data both in-sample, and out-of-sample using leave-one-out cross-validation.

TABLE 4 The FPL parameters are estimated at all weather stations, using the marginal model and the out-of-sample regression model

Model	$\hat{\lambda}$	Winter			Spring			Summer			Autumn		
		2.5%	50.0%	97.5%	2.5%	50.0%	97.5%	2.5%	50.0%	97.5%	2.5%	50.0%	97.5%
Marginal	$\hat{\lambda}_1$	3.0	5.0	7.7	3.6	8.0	11.5	3.7	8.8	11.8	3.1	5.6	7.4
	$\hat{\lambda}_2$	1.9	3.7	7.0	2.3	5.7	8.1	1.9	4.9	7.1	1.9	4.4	6.9
	$\hat{\lambda}_3$	-0.6	0.2	1.0	-0.9	0.1	1.0	-0.6	0.4	1.0	-0.7	0.6	1.0
	$\hat{\lambda}_4$	0.0	0.4	0.8	0.0	0.4	1.2	0.0	0.2	0.5	0.0	0.2	1.1
	$\hat{\lambda}_5$	-0.1	0.1	0.3	-0.2	0.2	0.5	-0.1	0.3	0.6	0.0	0.2	0.5
Regression	$\hat{\lambda}_1$	2.9	4.8	7.1	3.4	7.8	12.5	4.6	8.3	11.2	3.3	5.5	7.6
	$\hat{\lambda}_2$	1.5	3.8	6.2	2.3	5.7	8.7	3.6	5.2	7.1	1.9	4.6	6.7
	$\hat{\lambda}_3$	-0.5	0.2	0.9	-0.5	0.5	1.0	0.0	0.6	0.9	-0.3	0.5	0.8
	$\hat{\lambda}_4$	0.0	0.4	0.6	0.0	0.3	0.7	0.0	0.1	0.5	0.0	0.4	0.8
	$\hat{\lambda}_5$	-0.3	0.0	0.3	-0.3	0.2	0.7	0.0	0.3	0.7	-0.2	0.2	0.4

Note: Median parameter estimates over all 112 locations are displayed, along with the 2.5% and the 97.5% quantiles.

to have zero mean and a standard deviation of one. Modeling is performed in-sample using all available data, and out-of-sample in a leave-one-out cross-validation study. Thus, in the cross-validation study, the regression coefficients β_{p_i} , $i = 1, 2, \dots, 99$, are estimated 112 times for each season, each time by leaving one station out of the training data. The FPL parameters are then estimated at the one station that was not included in training the quantile regression models. Table 3 shows little difference in performance between in-sample and out-of-sample estimation, indicating that our model does not overfit to the data. For winter and autumn data, the differences between the CRPS of the marginal model and the regression model are small. However, during summer and spring, there is a considerable difference in performance between the two models. The calibration of the regression model is clearly worse than that of the marginal model for all seasons. However, PIT errors with a magnitude of 10^{-2} is still good, even though it is worse than a magnitude of 10^{-4} . All estimated regression coefficients for the 112 out-of-sample median regressions are displayed in Figure 7. The variability in $\hat{\beta}_{0.5}$ within each season is small, indicating that the estimation procedure is robust against minor changes in the training data. We see that the most influential explanatory variables across all seasons are the distance to the open sea and the historical temperature observation. There is a large difference between summer, spring and the other two seasons

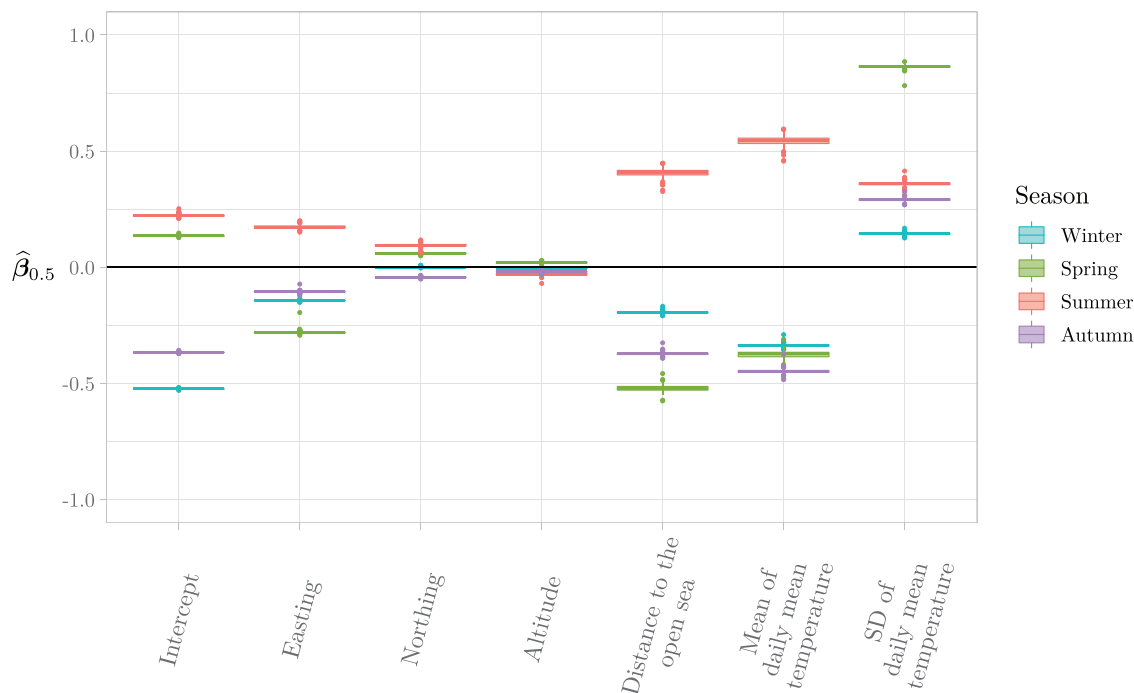


FIGURE 7 Estimated regression coefficients $\hat{\beta}_{0.5}$ for the median regression. The parameter estimation is performed out-of-sample, resulting in 112 parameter estimates within each season. The whiskers in the box-plot have a maximum length of 1.5 times the interquartile range. The y-axis has unit “standard deviations of \mathbf{y} ,” meaning that if a regression coefficient is for example, 0.4, then a change of one standard deviation for the corresponding explanatory variable causes a change of $0.4 \cdot \text{SD}(\mathbf{y})$ in the median of \mathbf{y}

for these explanatory variables. For example, a larger mean temperature will lead to larger median range during summer, but lower median range during all other seasons. This might be connected to the difference in model performance during summer and spring. Similar trends are found in the $\hat{\beta}_p$ for all other probabilities p .

Figure 8 displays the out-of-sample estimation results for the same stations and seasons as in Figure 6. While the regression model is able to capture the overall shapes at each location, some deviations are noticeable. In particular, the estimated right tail is too heavy at all stations but Hovden-Lundane, where it is too light. Apart from this, the model fits seem adequate for the bulk of the data. As seen in Table 4, the estimated FPL parameters from the out-of-sample regression model shares many similarities with the parameter estimates from the marginal model. Most of the spatial trends from the marginal model fits are also found when examining the estimates from the regional model.

Figure 9 summarizes the calibration of both the marginal model fits and the out-of-sample regression model fits for summer and winter seasons by displaying the PIT errors e_μ and e_σ at all locations. For the out-of-sample assessment of the regression model, the calibration of the estimated distributions varies substantially between the two seasons. In winter, the calibration is, expectedly, somewhat worse than that of the marginal model. However, only a few stations show considerable lack of calibration. Both positive and negative values of e_μ are observed, indicating both negative and positive biases. However, the values of e_σ are rather negative than positive, indicating a slight tendency towards overdispersion or too large spread. Similar patterns are observed for autumn (results not shown). The performance of the regression model in summer is considerably worse than that in winter. There is a distinct jump in the values of e_μ and e_σ as the distance from the sea increases. The jump in e_μ implies that we observe mostly positive biases along the coast and negative biases at inland locations. Figure 1 shows that there is a similar pattern of change in the median of diurnal temperature range along the coast and further inland during summer and spring. This indicates that the regression model is too smooth, and fails to model the transition from coastal to inland climate. A similar jump in e_μ and e_σ can be seen for spring data, although the difference is not as considerable as for the summer data.

Figure 3 hints that there is more information to gain from the climatological explanatory variables describing the distribution of mean temperature in winter and autumn than in summer and spring. In order to investigate whether the seasonal difference in skill is due to this effect, we repeated the regression analysis using only the four geographic and orographic explanatory variables. Under this model, the performance for winter and autumn data is considerably worse,

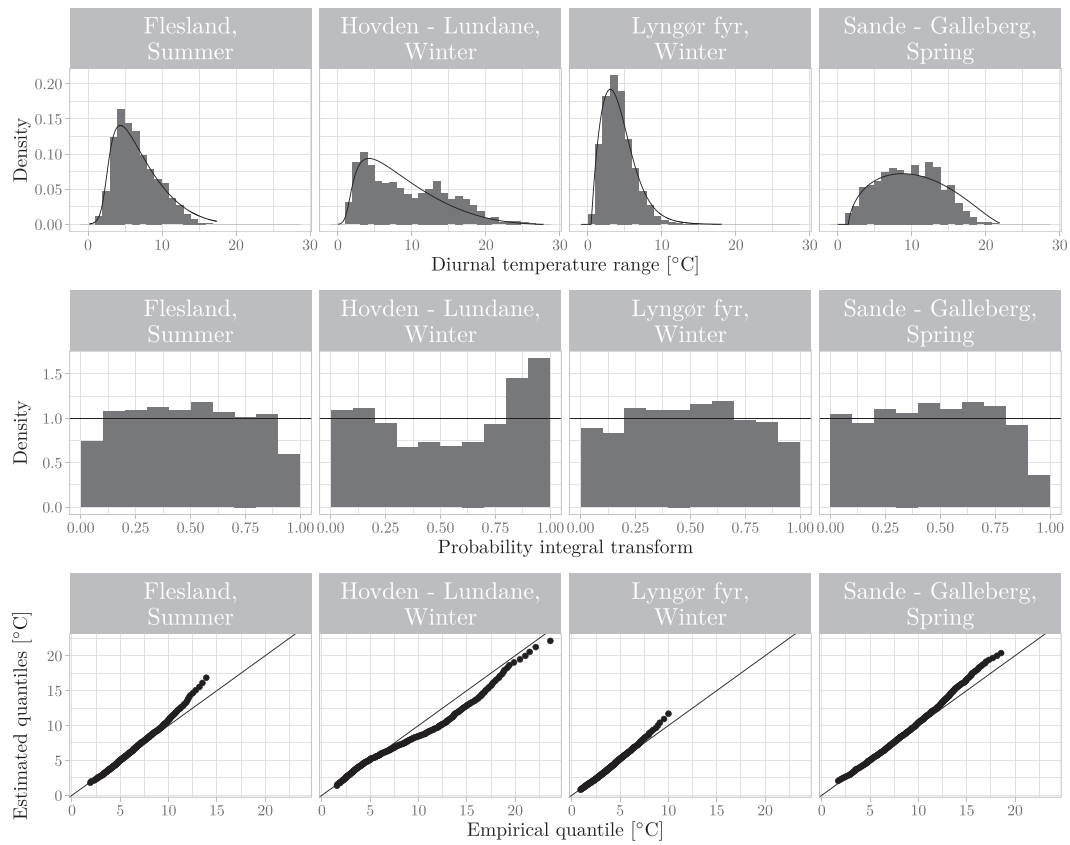


FIGURE 8 Upper plots: Histograms displaying observed diurnal temperature range are plotted along with the probability density functions of the FPL distribution from the out-of-sample regression model. Middle plots: Histograms displaying the PIT of diurnal temperature range with the estimated FPL parameters. Lower plots: QQ-plots displaying sample quantiles of diurnal temperature range against distributional quantiles of the estimated FPL distributions

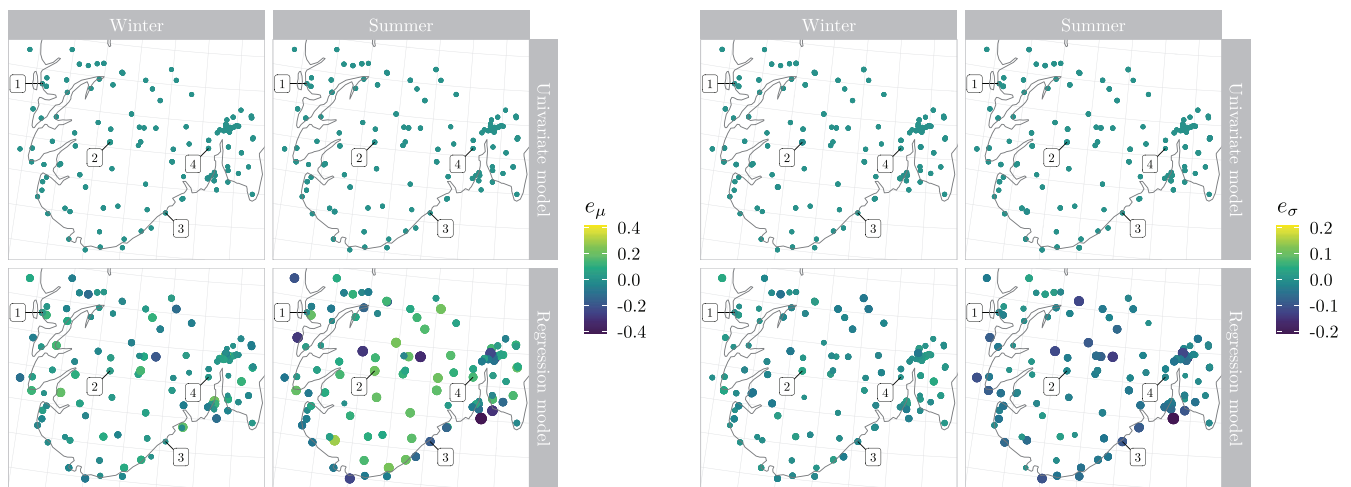


FIGURE 9 PIT mean errors e_μ and standard deviation errors e_σ are displayed for the regression model and the marginal model, during summer and winter. The magnitude of each error is represented by the radius of each dot. Values of each error are represented by the color of each dot. The locations of the four stations from Figures 2,6 and 8 are presented in all the plots. These are: (1) Flemland, (2) Hovden-Lundane, (3) Lyngør fyr, and (4) Sande-Galleberg

while it stays almost unchanged for spring and summer data (results not shown). The magnitude of the PIT errors during winter and autumn with this model is also comparable with those for spring and summer data in the original model.

7 | CONCLUSION AND DISCUSSION

This article proposes to use the FPL distribution to model the distribution of diurnal temperature range. A distributional quantile regression model is also proposed, where diurnal temperature range is modeled using a combination of quantile regression and marginal modeling with the FPL distribution. Parameter estimation is performed using the method of quantiles, which is a fast inference method that compares well with the competing inference methods of maximum likelihood estimation and the starship method. Diurnal temperature range from southern Norway is modeled using the marginal FPL model and the regression model. The marginal FPL model provides a good fit to diurnal temperature range data, and the regression model shows promise and is able to capture much, but not all, of the spatial trends in the distribution of diurnal temperature range.

We propose a reparametrization of the FPL distribution that allows for easier inference by directly connecting the location λ_1 and scale λ_2 to quantiles of the distribution. This allows us to select better initial values for parameter estimation of the FPL parameters. The reparametrization also limits the parameter space of λ so that $\lambda_5 > -1/2$. This does not appear to influence our results.

Although the method of quantiles shows great success in fitting the FPL distribution to data, we experience situations where the estimators for λ_4 and λ_5 become too large with respect to the true values, which the method accounts for by providing biased estimates for λ_3 . For large sample sizes this does not seem to affect the model fit much, but for smaller sample sizes it seems to negatively affect the performance. Further work should be put into understanding the effect of the parameters of the FPL distribution, and why the method of quantiles overestimates tail parameters in certain situations.

Model evaluation is mainly performed using the probability integral transform (PIT) and the continuous ranked probability score (CRPS). A novel closed-form expression for the CRPS with an FPL forecast distribution is developed, which makes model evaluation faster and simpler. Other scoring rules may be more appropriate for evaluating model fit in larger and more inhomogeneous regions (Bolin & Wallin, 2019), but we believe the CRPS to be a good choice for model evaluation in the current setting.

While the regression model is not able to fully capture the spatial patterns in diurnal temperature range for spring and summer, its performance is promising, especially for winter and autumn data. It is our belief that one can achieve better results with an improved selection of explanatory variables and, potentially, further development of the distributional quantile regression method. As an example, many of the weather stations with high PIT errors e_μ and e_σ are located along the coast. Consequently, it might be reasonable to better distinguish between coastal observations and inland observations, for example, through the introduction of a binary explanatory variable. A transformation of variables could also improve our models. The relationships between the median of diurnal temperature range and the available explanatory variables shown in Figure 3 do not seem linear for the geographical information. One might find possible transformations of the explanatory variables which are able to improve the linear relationships between the quantiles of diurnal temperature range and the available explanatory variables. In addition to the explanatory variables used in the current study, one might find important dependencies between diurnal temperature range and other climate variables, such as daily precipitation, wind speed and the degree of cloud cover. Especially precipitation and cloud cover have been found to be highly negatively correlated with diurnal temperature range (Waqas & Athar, 2018; Zhou et al., 2009). It is not obvious how such explanatory variables should be incorporated in our model, though. The regression model for diurnal temperature range only includes spatial fixed effects. The inclusion of spatial random effects might improve model performance (Lum & Gelfand, 2012; Self et al., 2021). Additionally, attempts to estimate parameter uncertainty will be affected by the temporal autocorrelation of diurnal temperature range, which is statistically significant for lags up to approximately one week. Consequently, further modeling attempts should also aspire to include a temporal framework. A comprehensive modeling framework for temperature range should furthermore include modeling components to account for data issues such as data inhomogeneities and measurement errors, including recording precision. This is particularly important in settings where long data series from various data sources are combined into a single analysis, like in the unified Bayesian approach that was introduced in the EUSTACE project (Rayner et al., 2020). In this project temporal and spatial autocorrelations were handled by latent random field components, and the data inhomogeneities were directly estimated via independent random effect variables. That approach allowed the propagation of uncertainty through the entire analysis system, in contrast to more traditional data homogenization methods that handle this as a separate pre-processing

step. In principle, the FPL model for diurnal temperature range can be incorporated into the observation level of such hierarchical models.

Two of the explanatory variables in the regression model are based on historical daily mean temperature. For spatial interpolation one might argue that observations of daily mean temperature are unavailable at most locations where there are no observations of diurnal temperature range. However, while the literature on modeling diurnal temperature range is lacking, much effort and success has been put into the modeling of mean temperature (e.g. Haylock et al., 2008; Maraun & Widmann, 2018). We assume that there already exist satisfactory spatial and temporal models for Norway, which are able to describe the historical mean and variance of daily mean temperature between 1989 and 2018 with a high performance. The Nordic Gridded Climate Data Set version 2 (Lussana et al., 2018), for example, models daily mean temperature with high performance everywhere in Norway, Finland, and Sweden. Accordingly, all explanatory variables can be provided at any location in Norway.

Better models for diurnal temperature range may be important for improving interpolation and statistical downscaling of temperature projections from climate models (e.g. Maraun & Widmann, 2018). The common approach today is to perform separate modeling of daily maximum, minimum, and mean temperature. However, this can lead to inconsistencies such as predictions where the daily mean is larger than the daily maximum temperature (e.g. Lussana et al., 2019). In addition, the three temperature variables are heavily dependent and should be modeled jointly, but multivariate modeling is often too challenging and computationally demanding. An alternative method for modeling these three temperature variables is to transform minimum, maximum, and mean temperature to diurnal temperature range, mean temperature, and the location of the daily mean inside the temperature range. This would remove all ordering inconsistencies between minimum, mean, and maximum temperature, and it would considerably reduce correlations between the three variables. Analysis of the data used in this article finds that the absolute values of pairwise sample correlations between daily minimum, maximum, and mean temperature mainly lie in the interval [0.9, 1]. Sample correlations between diurnal temperature range, daily mean temperature and the location of the mean inside the range, on the other hand, are always below 0.5. Thus, appropriate statistical models for diurnal temperature range and its relationship with daily mean temperature can be of great interest as model components for multivariate temperature modeling approaches. Further work should be conducted to examine this approach.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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